Dispersion Analysis of a TLM Mesh Using a New Scattering Matrix Formulation

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Abstract—**An equivalent scattering matrix for the TLM symmetrical condensed node is derived by rearranging the order of node ports. The new matrix is given in a partitioned form with zero blocks on the main diagonal. It enables a transformation of the general dispersion relation from a 12th- to a 6th-order eigenvalue equation, thus significantly simplifying the problem of finding a closed algebraic form of the dispersion relation for the symmetrical condensed node.**

I. Introduction

The SCATTERING matrix of the symmetrical condensed node (SCN) was derived originally by Johns [1] where the order of twelve node ports is arranged in an apparently arbitrary manner. Although the scattering matrix is a symmetric, unitary and sparse matrix, the original choice of node port numbering does not allow it to be partitioned and written in a compact form. Recently, a more systematic node numbering scheme was proposed in [2].

The form of the scattering matrix [1] does not present an important issue in the implementation of the scattering procedure since the use of scattering equations, rather than the scattering matrix itself, is recommended for higher efficiency [3]. Furthermore, a recent new approach for deriving scattering equations based on node voltages and loop currents [4] does not use the scattering matrix at all. However, manipulation of the scattering matrix is necessary when investigating the dispersion behavior of the node [5], where an eigenvalue problem with matrices of dimension $n = 12$ has to be solved. It would be convenient to represent the scattering matrix in a form which reduces the eigenvalue problem to a lower order.

In this paper, it is shown that an equivalent scattering matrix for the SCN, derived by rearranging the order of node ports, enables the general dispersion relation to be elegantly expressed in terms of matrices of size 6×6. The reduced size of the equivalent eigenvalue equation means that it can be more readily handled by present day symbolic packages and leads to a closed form expression for the dispersion relation. This is very useful in establishing trends and comparing the dispersion in different types of mesh. Thus far, most investigations of dispersion were numerical and only very recently closed form expressions have been produced [5]. The current paper describes a straightforward and much simpler approach for obtaining a compact dispersion relation.

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TABLE I
Order of Node Ports in the Scattering Matrix

New row/column position	Voltage index	Original port notation [1]
1	xny	3
2	ynz	5
3	znx	2
4	xpy	11
5	ypz	7
6	zpx	9
7	xnz	6
8	ynx	1
9	zny	4
10	xpz	10
11	ypx	12
12	zpy	8

II. New Form of Scattering Matrix

The notation of voltages in the SCN is given in the form V_{isj} where i denotes direction and j polarization of the appropriate transmission line ($i, j \in \{x, y, z\}$ and $i \neq j$), while $s \in \{n, p\}$ indicates the position of the port on the negative (n) or positive (p) side assuming the origin of coordinates is at the center of the node. The order of node ports for the new matrix representation is given in Table I. The original scheme numbering [1] is given in the third column for reference.

Rearranging rows and columns of the original scattering matrix [1] according to Table I gives

$$\mathbf{S} = \begin{bmatrix} 0 & \mathbf{S}_0 \\ \mathbf{S}_0^T & 0 \end{bmatrix} \tag{1}$$

where

$$\mathbf{S}_0 = \begin{bmatrix} \mathbf{S}_1 & \mathbf{S}_2 \\ \mathbf{S}_2 & \mathbf{S}_1 \end{bmatrix} \tag{2}$$

with

$$\mathbf{S}_1 = \frac{1}{2}\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \tag{3}$$

and

$$\mathbf{S}_2 = \frac{1}{2}\begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix}. \tag{4}$$

III. Representation of the General Dispersion Relation

A general dispersion relation for TLM nodes can be expressed in the form [5]:

$$\det(\mathbf{PS} - e^{jk_0 d}\mathbf{I}) = 0 \tag{5}$$

where k_0 is the propagation constant along the transmission lines, d is the node spacing, $\mathbf{I}$ is identity matrix and $\mathbf{S}$ is the scattering matrix (1), while $\mathbf{P}$ is a connection matrix. For the new arrangement of node ports, matrix $\mathbf{P}$ can be written in the form:

$$\mathbf{P} = \begin{bmatrix} \mathbf{P}_0 & 0 \\ 0 & \mathbf{P}_0 \end{bmatrix} \tag{6}$$

where

$$\mathbf{P}_0 = \begin{bmatrix} 0 & \mathbf{P}_1 \\ \mathbf{P}_1^* & 0 \end{bmatrix} \tag{7}$$

with

$$\mathbf{P}_1 = \begin{bmatrix} X & 0 & 0 \\ 0 & Y & 0 \\ 0 & 0 & Z \end{bmatrix} \tag{8}$$

and

$$X = e^{jk_x d} \qquad Y = e^{jk_y d} \qquad Z = e^{jk_z d}.$$

$\mathbf{P}_1^*$ stands for the Hermitian transpose of $\mathbf{P}_1$ and k_x, k_y, k_z are the unknown components of the mesh propagation vector.

Solution of the general dispersion relation (5) can be found by finding eigenvalues $\lambda_i = e^{jk_i d}$ of the matrix $\mathbf{PS}$ which is of order 12×12. The eigenvalue equation can be written in the form:

$$\mathbf{PS}\vec{X}_i = \lambda_i \vec{X}_i. \tag{9}$$

Using the partitioned forms of matrices $\mathbf{S}$ (1) and $\mathbf{P}$ (6) and by partitioning the eigenvector $\vec{X}_i = [\mathbf{X}_{1i}\mathbf{X}_{2i}]^T$ we can write (9) in the form:

$$\begin{bmatrix} 0 & \mathbf{P}_0\mathbf{S}_0 \\ \mathbf{P}_0\mathbf{S}_0^T & 0 \end{bmatrix} \begin{bmatrix} \mathbf{X}_{1i} \\ \mathbf{X}_{2i} \end{bmatrix} = \lambda_i \begin{bmatrix} \mathbf{X}_{1i} \\ \mathbf{X}_{2i} \end{bmatrix} \tag{10}$$

which leads to a system of two matrix equations:

$$\mathbf{P}_0\mathbf{S}_0\vec{X}_{2i} = \lambda \vec{X}_{1i} \tag{11}$$
$$\mathbf{P}_0\mathbf{S}_0^T\vec{X}_{1i} = \lambda_i \vec{X}_{2i}. \tag{12}$$

Combining these two equations we obtain:

$$\mathbf{P}_0\mathbf{S}_0\mathbf{P}_0\mathbf{S}_0^T\vec{X}_{1i} = (\lambda_i)^2\vec{X}_{1i}. \tag{13}$$

This equation has the form of an eigenvalue equation for a matrix $\mathbf{S}' = \mathbf{P}_0\mathbf{S}_0\mathbf{P}_0\mathbf{S}_0^T$ with eigenvectors $\vec{X}_{1i}$ and eigenvalues $\eta_i = (\lambda_i)^2$, requiring:

$$\det(\mathbf{S}' - \eta_i\mathbf{I}) = 0. \tag{14}$$

Therefore, by solving the eigenvalue equation (14) of the 6th order for η_i we can obtain eigenvalues λ_i of the 12×12 matrix $\mathbf{PS}$ as $\lambda_i = \pm\sqrt{\eta_i}$.

IV. Solution of the Dispersion Relation

Having reduced the size of eigenvalue problem (5) with 12×12 matrices to the equivalent one (14) with 6×6 matrices, its analytical solution is relatively simple. The six eigenvalues η_i of the matrix $\mathbf{S}'$ can be found from (14) as:

$$\eta_1 = \eta_2 = 1 \tag{15}$$
$$\eta_3 = \eta_4 = C_1 + \sqrt{C_1^2 - 1} \tag{16}$$
$$\eta_5 = \eta_6 = C_1 - \sqrt{C_1^2 - 1} \tag{17}$$

where

$$C_1 = \frac{1}{2}\left(\sum_{k_p, k_q} \cos(k_p d)\cos(k_q d) - 1\right) \tag{18}$$

and $(k_p, k_q) \in \{(k_x, k_y), (k_y, k_z), (k_z, k_x)\}$.

(15) represents nonpropagating stationary solutions. Since $\eta_3 = \eta_5^{-1}$, equation (16) and (17) represent solutions for waves propagating in positive and negative directions, respectively. It is interesting to note that solutions (15)–(17) have a similar form to the first three solutions for the 2D TLM [5] and the scalar 3D TLM nodes [6], the only difference being in the definition of the constant C_1. However, eigenvalue η_i are not the eigenvalues of matrix $\mathbf{PS}$, but of the equivalent matrix $\mathbf{P}_0\mathbf{S}_0\mathbf{P}_0\mathbf{S}_0^T$. The correct eigenvalue λ_i are calculated as square roots of η_i.

Grouping the reciprocal solutions (16) and (17) together and substituting for $k_i = k_0$, the dispersion relation for propagating modes can be written in the form:

$$\cos^2(k_0 d) = \tfrac{1}{2}(C_1 + 1) \tag{19}$$

where C_1 is as defined in (18). This formula is the same as in [5] but derived in a more compact manner.

V. Conclusion

A new formulation of the scattering matrix for the TLM SCN node was presented. Its compact definitions was used to facilitate reduction of the eigenvalue problem related to the general dispersion relation for the SCN. It was shown that an equivalent eigenvalue equation involving matrices of reduced size (6×6) can be used to find the squares of all eigenvalue solutions of the original matrix (of size 12×12), leading to an elegant expression of the dispersion relation in a closed algebraic form.

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